Games and Economic Behavior ••• (••••) •••–•••

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Games and Economic Behavior

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Note

The Herodotus paradox ☆

Michael R. Baye [a,*], Dan Kovenock [b], Casper G. de Vries [c]

[a] Department of Business Economics and Public Policy, Kelley School of Business, Indiana University, Bloomington, IN 47405, United States
[b] Economic Science Institute, Chapman University, Orange, CA, United States
[c] Tinbergen Institute and Erasmus University Rotterdam, Netherlands

ARTICLE INFO

Article history:
Received 19 July 2010
Available online xxxx

JEL classification:
C72
D44

Keywords:
Auctions
Contests
Mixed strategy
Interim equilibrium
Ex ante equilibrium
Fubini's Theorem

ABSTRACT

The Babylonian bridal auction, described by Herodotus, is regarded as one of the earliest uses of an auction in history. Yet, to our knowledge, the literature lacks a formal equilibrium analysis of this auction. We provide such an analysis for the two-player case with complete and incomplete information, and in so doing identify what we call the "Herodotus paradox."

1. Introduction

Cassady (1967) and Milgrom and Weber (1982) note that Herodotus' account of 500 B.C. Babylonian bridal auctions is one of the earliest recorded uses of an auction in history. According to the translation by Rawlinson (1885),[1] Herodotus wrote:

> "Once a year in each village the maidens of age to marry were collected… Then a herald called up the damsels one by one, and offered them for sale. He began with the most beautiful… when the herald had gone through the whole number of the beautiful damsels, he should then call up the ugliest… and offer her to the men, asking who would agree to take her with the smallest marriage-portion… The marriage-portions were furnished by the money paid for the beautiful damsels…"

We examine a simultaneous-move (sealed bid) version of Herodotus' Babylonian bridal auction with two suitors and two maidens. Suitor $i \in \{1, 2\}$ values the more beautiful of the two maidens at $v_i^B > 0$ and the less attractive maiden at $v_i^L < v_i^B$. The higher bidder wins the more beautiful maiden, and pays the auctioneer the amount bid by the lower bidder. The

☆ We have benefited from the insightful comments of an anonymous referee and the Associate Editor. We are grateful to Chaim Fershtman, Vladimir Karamychev, Michael Rauh, and Eric Schmidbauer for helpful conversations.

* Corresponding author.
E-mail address: mbaye@indiana.edu (M.R. Baye).

[1] An online version of this translation by Rawlinson is available in Chapter 196 at the following website: http://www.shsu.edu/~his_ncp/Herobab.html.

0899-8256/$ – see front matter
doi:10.1016/j.geb.2011.07.004

auctioneer then transfers this entire amount to the lower bidder as a "sweetener" along with the less fair of the two maidens. Notice that, as in Herodotus' original account, the "sweetener" is a transfer from the winner to the loser, such that the auctioneer's revenues are zero in the mechanism.

The payoff to suitor i when he bids $x_i \in [0, \infty)$ and suitor j bids $x_j \in [0, \infty)$ is:

$$u_i\big(x_i, x_j; v_i^B, v_i^L\big) = \begin{cases} v_i^B - x_j & \text{if } x_i > x_j \\ \frac{1}{2}(v_i^B - x_j + v_i^L + x_i) & \text{if } x_i = x_j \\ v_i^L + x_i & \text{if } x_i < x_j \end{cases} \tag{1}$$

We assume that in the case of a tie the auctioneer allocates the maidens to the bidders based on the flip of a fair coin. Notice that in the case of complete information and symmetric values ($v_i^B = v_B > v_i^L = v_L$), this is a constant-sum game.

In Section 2 we examine this auction with complete information. We show that the standard technique used to construct symmetric mixed-strategy equilibria (which involves identifying an atomless mixed strategy in which each player's expected payoff is constant on its support and verifying that a player cannot improve his payoff by submitting a bid outside of the support) yields an apparent paradox: There exists a continuum of symmetric mixed-strategy equilibria in which each player earns an arbitrarily high payoff, despite the fact that the game is constant-sum and bids are pure transfers between the two players. We show that this paradox stems from the fact that the standard techniques for constructing mixed-strategy equilibria in games of complete information are not sufficient to guarantee that the resulting mixed strategy is a symmetric Nash equilibrium. In particular, the standard steps identify a symmetric mixed strategy, F^*, such that, for any bid contemplated by one player, his expected payoff is constant on its support and cannot be improved by deviating to a bid outside of the support, so that x_i is a best response to F^*. Notice that if an outside arbiter assigns each player a pure strategy that is an independent draw from such an F^*, and each player's particular assignment is private information, then neither suitor has an incentive to unilaterally deviate from his assigned bid, as the expected payoff from submitting *any* such assigned bid cannot be improved by submitting any alternative bid. The paradox stems from the fact that this "assignment equilibrium" is not a Nash equilibrium because expected utility with respect to the joint distribution induced by F^* does not exist. In other words, even though F^* is an "assignment equilibrium," F^* is not a best response to F^* because players cannot determine their (*ex ante*) expected utility from employing F^* as a strategy.

More formally, the Herodotus paradox is related to the following question. Suppose S_1 and S_2 are separable metric spaces of pure strategies with associated sets of mixed strategies $\Delta(S_i)$ consisting of regular Borel probability measures, and let $u_i : S_i \times S_j \to \mathbb{R}$ denote the (measurable) payoff function of player i. Suppose that $\mu_i \in \Delta(S_i)$ and let $\mu_i \otimes \mu_j$ denote the associated product measure on the product Borel σ-field $\mathcal{B}(S_i) \otimes \mathcal{B}(S_j)$. Further suppose that, $\forall x_i \in S_i$:

$$\int\limits_{S_j} u_i(x_i, x_j)\, d\mu_j$$

exists and is finite, and that each μ_i is a mixed strategy that assigns all probability measure to the (nonempty set of) x_i's that are best responses (in S_i) to μ_j. Does it follow that

$$\int\limits_{S_i \times S_j} u_i(x_i, x_j)\, d(\mu_i \otimes \mu_j)$$

exists (even as an extended real)? Our analysis of the Babylonian bridal auction reveals that the answer is no, and hence, μ_i's satisfying these properties are not mutual best responses in the Babylonian bridal auction. Indeed, this is an example in which Fubini's Theorem does not hold: $\int_{S_i}[\int_{S_j} u_i(x_i, x_j)\, d\mu_j]\, d\mu_i \neq \int_{S_j}[\int_{S_i} u_i(x_i, x_j)\, d\mu_i]\, d\mu_j$.

We also show that, while nonexistence of expected utility in the Babylonian bridal auction stems from the unboundedness of payoffs, there are many economic games (including the war of attrition, the sad loser auction, and a variant of the Babylonian bridal auction) with unbounded payoffs that do not lead to existence problems. The primary take-away from our analysis is that it is important to add an additional step to the standard analysis for constructing mixed-strategy Nash equilibria in games with unbounded payoffs: One must verify that players' utilities are integrable with respect to the joint distribution of putative equilibrium mixed strategies. To the best of our knowledge, this point has not been identified in the literature; in fact, we stumbled upon it purely because of the Herodotus paradox.

Section 3 shows that similar issues arise in games of incomplete information, thereby demonstrating that the issues are not purely an artifact of mixed strategies. In auctions with incomplete information, the standard *five* step method used to derive symmetric pure strategies in fact yields *interim* equilibria, and these equilibria may not comprise an *ex ante* equilibrium because of the failure of *ex ante* expected utility to exist. We provide an example in the context of the Babylonian bridal auction that leads to the Herodotus paradox with incomplete information: Conditional on the maidens being unveiled (so that their values are private information to the suitors), there exists a continuum of (interim) equilibria in which players earn an arbitrarily high payoff—even though all moments of the assumed value distribution are finite. Yet, none of these paradoxical equilibria are *ex ante* equilibria. As before, the failure of these equilibria to constitute *ex ante* equilibria arises because the *ex ante* expected utility arising from the (paradoxical) interim equilibria cannot be computed in the first place.

2. The bridal auction with complete information

Suppose first that players are symmetric and have complete information, so that $v_i^B \equiv v_B$ and $v_i^L \equiv v_L$ in Eq. (1). One may readily verify that $x^* = (v_B - v_L)/2$ is a symmetric pure-strategy equilibrium, and that this is the unique symmetric pure-strategy equilibrium.[2] In this equilibrium, each suitor earns a payoff of $EU = \frac{1}{2}(v_B + v_L)$.

2.1. A paradox

The standard approach for finding a symmetric mixed-strategy equilibrium typically involves three steps: (1) identify a "candidate" continuous distribution function F such that each player's expected payoff is constant on its support, given that the rival's bid is determined by F, (2) verify that "candidate" F is indeed a well-defined continuous cumulative distribution function, and (3) show that neither player can unilaterally increase his payoff by submitting a bit outside of the support of F, given that the rival's bid is determined by F.

Following this approach, suppose the rival suitor bids according to an atomless F on $[m, u]$ (so that the probability of a "tie" is zero). Then the expected payoff to a player that submits a bid of $x \in [m, u]$ against his rival's F is

$$EU(x) = \int_m^x (v_B - s)\,dF(s) + \int_x^u (v_L + x)\,dF(s) = \int_m^x (v_B - s) f(s)\,ds + \int_x^u (v_L + x) f(s)\,ds$$

Using step (1) and letting f denote the density of F, constancy of expected payoffs requires that for all $x \in [m, u]$:

$$0 = \frac{d}{dx}EU(x) = (v_B - x) f(x) - (v_L + x) f(x) + \int_x^u f(s)\,ds = (v_B - v_L - 2x) f(x) + 1 - F(x)$$

which implies $v_B - v_L - 2x < 0$ for $x \in [m, u)$. The solution to this differential equation is

$$F(x) = 1 - \left(\frac{c}{v_B - v_L - 2x}\right)^{\frac{1}{2}}$$

where $c < 0$ is a constant determined in step (2).

Moving to step (2), for this to be a well-defined continuous distribution function requires $F(m) = 0$, which implies $c = v_B - v_L - 2m < 0$ (and hence, $m > (v_B - v_L)/2$). Next, setting $F(u) = 1$ implies $u = \infty$. Hence, the candidate symmetric equilibrium entails each suitor i submitting a bid, x_i, based on a cumulative distribution function

$$F^*(x_i) = 1 - \left(\frac{v_B - v_L - 2m}{v_B - v_L - 2x_i}\right)^{\frac{1}{2}} \quad \text{on } [m, \infty) \tag{2}$$

where $m \in (\frac{v_B - v_L}{2}, \infty)$ is arbitrary. To complete step (2), notice that F^* is a well-defined atomless probability distribution with density

$$f^*(x_i) = \frac{dF^*}{dx_i} = \left(\frac{v_B - v_L - 2x_i}{v_B - v_L - 2m}\right)^{\frac{1}{2}} \left(\frac{2m + v_L - v_B}{(v_B - v_L - 2x_i)^2}\right) > 0.$$

Turning to step (3), the expected payoff to a player that submits a bid of $x \in [m, \infty)$ against his rival's F^* is

$$EU(x) \equiv \int_m^x (v_B - s)\,dF^*(s) + \int_x^\infty (v_L + x)\,dF^*(s) = v_L + m \tag{3}$$

which is constant on $[m, \infty)$. Note that for $x' < m$, $EU(x') = v_L + x' < EU(m)$, which means a suitor cannot improve his payoff by submitting a bid below m, given that the rival's bid is based on F^*.

Thus, applying the usual reasoning—that is, steps (1) through (3)—one would conclude that F^* is a symmetric mixed-strategy Nash equilibrium in which each player earns a finite expected payoff of $EU^* = v_L + m < \infty$. Notice that, since $m > (v_B - v_L)/2$ is arbitrary, there is a continuum of such equilibria and, for arbitrarily large m, each player's payoff is arbitrarily large (but finite) in any such equilibrium. This is what we call the Herodotus paradox with complete information: The bid transferred to the loser exactly equals the amount paid by the winner (the auctioneer earns zero profit with probability one), so it would seem that sum of the two suitors' payoffs is constant $(v_B + v_L)$. Yet, application of steps (1) through (3) leads to the conclusion that players can earn an arbitrarily high expected payoff in a symmetric mixed-strategy Nash equilibrium. That is, the mixed strategies derived using standard arguments appear to lead to a "utility pump."

[2] Uniqueness follows from Proposition 1 in Baye et al. (forthcoming).

Please cite this article in press as: Baye, M.R., et al. The Herodotus paradox. Games Econ. Behav. (2011), doi:10.1016/j.geb.2011.07.004

2.2. A closer look at the paradox

The Herodotus paradox suggests that the standard arguments used to derive mixed-strategy equilibria may be incomplete. Notice that the standard arguments imply that if one suitor's bid is a random draw from F^*, then the other suitor is indifferent between submitting any bid $x \in [m, \infty)$ and strictly prefers such a bid to bidding $x' < m$. Based on this, it is tempting to conclude that—so long as one suitor randomizes based on F^*—the other suitor can do no better than to also choose a bid at random from F^*, since it places all mass on $[m, \infty)$. Using Eqs. (1) and (3), this reasoning would seem to imply that the equilibrium expected payoff to a player that randomizes against F^* by (independently) using F^* himself is (letting $\mu^*_{x_i,x_j}$ denote the product measure induced by $F^*(x_i)$ and $F^*(x_j)$)

$$\begin{aligned} u_i\big(F^*, F^*\big) &\equiv \int\limits_{[m,\infty)\times[m,\infty)} u_i\big(x_i, x_j; v_i^B, v_i^L\big)\, d\mu^*_{x_i,x_j} \\ &= \int\limits_m^\infty \int\limits_m^\infty u_i\big(x_i, x_j; v_i^B, v_i^L\big)\, dF^*(x_i)\, dF^*(x_j) \qquad (4) \\ &= \int\limits_m^\infty \int\limits_m^\infty u_i\big(x_i, x_j; v_i^B, v_i^L\big)\, dF^*(x_j)\, dF^*(x_i) \qquad (5) \\ &= \int\limits_m^\infty EU(x_i)\, dF^*(x_i) \\ &= v_L + m \end{aligned}$$

It turns out that this reasoning fails in the Babylonian bridal auction with complete information: F^* is not a best response to F^*—not because there is a profitable deviation, but because $u_i(F^*, F^*)$ does not exist. Consequently, the conditions of Fubini's Theorem (Chung, 1974, pp. 59–60) are not satisfied; indeed, in the case at hand the (Lebesgue–Stieltjes) integrals in Eqs. (4) and (5) are *not* equal.[3] In short, $u_i(F^*, F^*) \neq \int_m^\infty EU(x_i)\, dF^*(x_i)$, and it is erroneous to use the fact that $EU(x_i) = v_L + m$ is constant for $x_i \in [m, \infty)$ and $EU(x_i) < v_L + m$ for $x_i < m$ to conclude that F^* is a best response to F^*. The Herodotus paradox—that the sum of the putative equilibrium payoffs can exceed $v_B + v_L$ by an arbitrarily large amount—stems from the fact that the putative strategies do not comprise a Nash equilibrium in the first place.

To establish these assertions, first recall that the expectation of a random variable X does not exist if $E[X^+] = E[X^-] = \infty$, where $X^+ \equiv \max(0, X)$ and $X^- \equiv \max(-X, 0)$ (see Chung, 1974, p. 40). We will demonstrate that $u_i(F^*, F^*)$ does not exist by taking $X \equiv u_i(x_i, x_j; v_i^B, v_i^L)$ and showing that $E[X^+] = E[X^-] = \infty$.

Note that $E[X^+]$ and $E[X^-]$ both exist since they are, by definition, expectations of nonnegative real numbers. Hence, Fubini's Theorem (see Chung, 1974, p. 60) implies that the integral of X^+ and X^- with respect to the product measure induced by F^* can be written as a double integral that is invariant to the order of integration.[4]

Thus, for the case at hand,

$$\begin{aligned} E\big[X^+\big] \equiv E\big[\max(u_i, 0)\big] = {} & \int\limits_m^\infty \int\limits_m^\infty \max\big((v_B - x_j), 0\big) I_{x_j<x_i}\, dF^*(x_i)\, dF^*(x_j) \\ & + \int\limits_m^\infty \int\limits_m^\infty \max\big((v_L + x_i), 0\big) I_{x_i<x_j}\, dF^*(x_i)\, dF^*(x_j) \end{aligned}$$

The first term is finite, since the integrand is bounded above by v_B. The second term is

$$\begin{aligned} \int\limits_m^\infty \int\limits_m^\infty \max\big((v_L + x_i), 0\big) I_{x_i<x_j}\, dF^*(x_i)\, dF^*(x_j) &= \int\limits_m^\infty \int\limits_m^\infty (v_L + x_i) I_{x_i<x_j} f^*(x_i) f^*(x_j)\, dx_i\, dx_j \\ = v_L \int\limits_m^\infty \int\limits_m^\infty I_{x_i<x_j} f^*(x_i) f^*(x_j)\, dx_i\, dx_j &+ \int\limits_m^\infty \int\limits_m^\infty x_i I_{x_i<x_j} f^*(x_j) f^*(x_i)\, dx_j\, dx_i \end{aligned}$$

[3] Indeed, one can show that $\int_m^\infty \int_m^\infty u_i(x_i, x_j; v_i^B, v_i^L)\, dF^*(x_i)\, dF^*(x_j) = v_B - m$ while $\int_m^\infty \int_m^\infty u_i(x_i, x_j; v_i^B, v_i^L)\, dF^*(x_j)\, dF^*(x_i) = v_L + m$.

[4] The integrals below are interpreted as Lebesgue–Stieltjes integrals. Although Lebesgue–Stieltjes integration does not generally coincide with Riemann–Stieltjes integration, in the present case the two methods coincide. Thus, either method may be used to verify the steps below.

$$= v_L \int_m^\infty \int_m^\infty I_{x_i<x_j} f^*(x_i) f^*(x_j)\, dx_i\, dx_j + \int_m^\infty x_i f^*(x_i) \int_m^\infty I_{x_i<x_j} f^*(x_j)\, dx_j\, dx_i$$

$$= v_L \int_m^\infty \int_m^\infty I_{x_i<x_j} f^*(x_i) f^*(x_j)\, dx_i\, dx_j + \int_m^\infty x_i f^*(x_i)\bigl(1 - F^*(x_i)\bigr)\, dx_i$$

Once again the first term is finite. The second term is

$$\begin{aligned}
\int_m^\infty x_i f^*(x_i)\bigl(1 - F^*(x_i)\bigr)\, dx_i &= \int_m^\infty x_i \left(\frac{v_B - v_L - 2x_i}{v_B - v_L - 2m}\right)^{\frac{1}{2}} \left(\frac{2m + v_L - v_B}{(v_B - v_L - 2x_i)^2}\right)\left(\frac{v_B - v_L - 2m}{v_B - v_L - 2x_i}\right)^{\frac{1}{2}} dx_i \\
&= \int_m^\infty x_i \left(\frac{2m + v_L - v_B}{(v_B - v_L - 2x_i)^2}\right) dx_i \\
&= (2m + v_L - v_B) \int_m^\infty \frac{x_i}{(v_B - v_L - 2x_i)^2}\, dx_i \\
&= \infty
\end{aligned}$$

Hence, we conclude that $E[X^+] = \infty$. Similarly,

$$\begin{aligned}
E\bigl[X^-\bigr] \equiv E\bigl[\max(-u_i, 0)\bigr] &= \int_m^\infty \int_m^\infty \max\bigl(-(v_B - x_j), 0\bigr) I_{x_j<x_i}\, dF^*(x_i)\, dF^*(x_j) \\
&\quad + \int_m^\infty \int_m^\infty \max\bigl(-(v_L + x_i), 0\bigr) I_{x_i<x_j}\, dF^*(x_i)\, dF^*(x_j) \\
&= \int_m^\infty f^*(x_j) \max\bigl(-(v_B - x_j), 0\bigr) \int_m^\infty I_{x_j<x_i} f^*(x_i)\, dx_i\, dx_j \\
&= \int_m^\infty f^*(x_j) \max\bigl(-(v_B - x_j), 0\bigr)\bigl(1 - F^*(x_j)\bigr)\, dx_j \\
&= \int_m^{v_B} 0\, dx_j + \int_{v_B}^\infty (x_j - v_B)\bigl(1 - F^*(x_j)\bigr) f^*(x_j)\, dx_j \\
&= -v_B \int_{v_B}^\infty \bigl(1 - F^*(x_j)\bigr) f^*(x_j)\, dx_j + \int_{v_B}^\infty x_j \bigl(1 - F^*(x_j)\bigr) f^*(x_j)\, dx_j
\end{aligned}$$

The first term is once again bounded. The second term is

$$\begin{aligned}
\int_{v_B}^\infty x_j \bigl(1 - F^*(x_j)\bigr) f^*(x_j)\, dx_j &= \int_{v_B}^\infty x_j \left(\frac{v_B - v_L - 2m}{v_B - v_L - 2x_j}\right)^{\frac{1}{2}} \left(\frac{v_B - v_L - 2x_j}{v_B - v_L - 2m}\right)^{\frac{1}{2}} \left(\frac{2m + v_L - v_B}{(v_B - v_L - 2x_j)^2}\right) dx_j \\
&= \int_{v_B}^\infty x_j \left(\frac{2m + v_L - v_B}{(v_B - v_L - 2x_j)^2}\right) dx_j \\
&= \infty
\end{aligned}$$

Since $E[X^+] = E[X^-] = \infty$, $E[X] \equiv u_i(F^*, F^*)$ does not exist. Thus, F^* is not a symmetric Nash equilibrium to the Babylonian bridal auction, so the unique symmetric Nash equilibrium is $x^* = (v_B - v_L)/2$.

To summarize, existence of a symmetric mixed-strategy Nash equilibrium requires that F^* be an (*ex ante*) best response to F^*. The F^* constructed based on standard arguments (the three steps described above) does not satisfy this condition—not because there is a profitable deviation, but because the expectation cannot be computed in the first place. The larger point is that in auctions and contests with "spillovers" and payoff functions that are unbounded, it is important to add to steps (1) through (3) a fourth step: one must (4) verify that each player's utility is integrable with respect to the joint distribution of putative equilibrium mixed strategies—F^* in this case.[5] In the interest of full disclosure, we stumbled upon the relevance of the fourth step purely because of the paradox; had F^* not led to a paradox, we would have presumed it was a legitimate equilibrium to the Babylonian bridal auction.

It is important to stress that, in games where payoff functions are bounded, if steps (1) through (3) are satisfied then step (4) is automatically satisfied. However, there are a number of important games in economics where payoff functions are unbounded and steps (1) through (3) lead to a symmetric mixed strategy with unbounded support. Probably the best known example is the war of attrition, where the analogue of Eq. (1) is

$$u_i(x_i, x_j; v) = \begin{cases} v - x_j & \text{if } x_i > x_j \\ \frac{1}{2}(v - x_j - x_i) & \text{if } x_i = x_j \\ -x_i & \text{if } x_i < x_j \end{cases}$$

and the analogue to Eq. (2) is

$$F^*_{War}(x) = 1 - \exp(-x/v) \quad \text{on } [0, \infty)$$

Notice, however, that with $X \equiv u_i(x_i, x_j; v)$, $E[X^+] < \infty$ and $E[X^-] < \infty$. Hence in the war of attrition, utility is integrable with respect to the joint distribution induced by F^*_{War}. In other words, step (4) is satisfied in the war of attrition, so F^*_{War} is indeed a symmetric mixed-strategy Nash equilibrium.

As a second example, consider a variant of the Babylonian bridal auction in which the low bidder receives, in addition to the transfer from the winning bidder, a matching dowry from an outside party. In this case,

$$u_i(x_i, x_j) = \begin{cases} v_B - x_j & \text{if } x_i > x_j \\ \frac{1}{2}(v_B - x_j + v_L + 2x_i) & \text{if } x_i = x_j \\ v_L + 2x_i & \text{if } x_i < x_j \end{cases} \tag{6}$$

One may directly verify that $x^* = (v_B - v_L)/3$ is a symmetric pure-strategy equilibrium. In addition, there also exists a continuum of "candidate" symmetric mixed-strategy equilibria that satisfy steps (1) through (3), which is as follows: For every $m \in (\frac{v_B - v_L}{3}, \infty)$,

$$F^*_{Dow}(x) = 1 - \left(\frac{v_B - v_L - 3m}{v_B - v_L - 3x}\right)^{\frac{2}{3}} \quad \text{on } [m, \infty)$$

Straightforward calculations reveal that, since $X \equiv u_i(x_i, x_j)$, $E[X^+] < \infty$ and $E[X^-] < \infty$. Thus, step (4) is also satisfied. It follows that F^*_{Dow} is a symmetric mixed-strategy equilibrium in which each player earns an expected payoff of $EU^* = 2m + v_L \in (\frac{2}{3}v_B + \frac{1}{3}v_L, \infty)$. Since $m > (v_B - v_L)/3$ is arbitrary, there is a continuum of such equilibria in which each player's expected payoff is arbitrarily large. However, the Babylonian bridal auction with a matching dowry does not exhibit a Herodotus paradox, since these arbitrarily high payoffs come out of the hide of the outside party that pays the matching dowry. Interestingly, for any given $m \in (\frac{v_B - v_L}{3}, \infty)$, $\int_m^\infty x\, dF^*_{Dow} = \infty$, so that the expected bid of each player is unbounded. This illustrates that unbounded expected bids do not imply the failure of a candidate equilibrium that satisfies steps (1) through (3) to, in fact, be a Nash equilibrium.[6]

2.3. Discussion

This analysis illustrates that there are games where mixed strategies satisfy the standard three steps (constancy of payoffs on the support, well-defined probability distribution, and no profitable deviation outside of the support) but yet do not comprise a Nash equilibrium. Nonetheless, in these instances a mixed strategy constructed using steps (1) through (3) may be viewed as an *assignment* equilibrium: If an outside arbiter assigns each player a pure strategy that is an independent draw from a mixed strategy satisfying steps (1) through (3) and each player's particular assignment is private information, then given the assigned pure strategy each player's expected payoff exists and is given by $EU(x_i)$.

For the case of the Babylonian bridal auction with complete information, if an outside arbiter assigns each suitor a bid x_i, it is common knowledge that each player's bid assignment is an independent draw from F^*, and each suitor's particular assignment is private information, then given the assigned bid each suitor's expected payoff exists and is given by $EU(x_i) = m + v_L \in (\frac{1}{2}v_B + \frac{1}{2}v_L, \infty)$. Moreover, neither suitor has an incentive to deviate from x_i, as the expected payoff from submitting *any* such assigned bid is constant on $[m, \infty)$ and cannot be improved by submitting any alternative bid. Thus,

[5] A random variable is integrable if its expectation exists and is finite; see Chung (1974, p. 40). If one is willing to admit equilibria in which players earn infinite equilibrium payoffs, one can replace this fourth step with a step that merely verifies existence of expected utility with respect to the product measure induced by F^*.

[6] The complete information version of the Riley–Samuelson sad loser auction is another example of a game that has a symmetric mixed-strategy Nash equilibrium where players' expected bids are unbounded.

steps (1) through (3) essentially yield an equilibrium to this assignment game. The equilibrium to this assignment game is not a Nash equilibrium to the original game, however, since players cannot compute their (*ex ante*) utility from participating in this assignment game. Indeed, the Herodotus paradox is related to a similar observation made by Bhattacharyya and Lipman (1995) in the context of a speculative bubble game. They show that two symmetric traders can enjoy positive expected gains to exchange, despite the fact that the gain to one trader in any realization exactly equals the loss to the other trader. Similar to the Herodotus paradox, this result stems from the fact that the underlying game does not have an *ex ante* equilibrium (owing to the nonexistence of *ex ante* expected utility), but does have an *interim* equilibrium.

In contrast, the symmetric equilibrium mixed strategies in the war of attrition and the Babylonian bridal auction with a matching dowry satisfy steps (1) through (4), so these mixed strategies are both an equilibrium to the original game (the mixed strategies are mutual best responses) and an equilibrium to the assignment game (any x_i assigned is a best response to the rival's mixed strategy).

3. The bridal auction with incomplete information

The paradox identified above is neither an artifact of mixed strategies nor the assumption that the players have complete information. To see this, suppose that it is common knowledge that both suitors value the lesser maiden at $v_i^L \equiv 0$, but that the suitors' valuations of the fairer maiden are privately observed random variables, $v_i^B \equiv v_i$, which are independently and identically distributed with an exponential distribution function, $G(v) = 1 - \exp(-v)$ on $[0, \infty)$ with associated density $g(v) = \exp(-v)$. Note that $E[v^k] = (k-1)!$ for all $k = 1, 2, 3, \ldots$. Hence, all moments of the assumed value distribution are bounded, including $E[v] = 1$. Thus, each suitor's (*ex ante*) expected value of the most beautiful of the two maidens is unity.[7] Under these assumptions, Eq. (1) simplifies to

$$u_i(x_i, x_j; v_i) = \begin{cases} v_i - x_j & \text{if } x_i > x_j \\ \frac{1}{2}(v_i - x_j + x_i) & \text{if } x_i = x_j \\ x_i & \text{if } x_i < x_j \end{cases}$$

The standard approach to solving for a symmetric (pure-strategy) equilibrium in this incomplete information environment is to: (1) assume that the rival follows a monotonically increasing bid function, $x_j(v_j)$, that maps the rival's valuation into a bid; (2) determine the bid, x_i, that solves the first-order condition for maximizing i's expected payoff given that he knows his own valuation is v_i but not the specific valuation of the rival; (3) impose symmetry of the two players' bid functions and solve for a candidate symmetric equilibrium bid function; (4) verify that it is monotonically increasing; and (5) verify that a player cannot profitably deviate from the symmetric bid function.

In the case at hand, step (1) implies that the expected payoff to suitor i who knows his own valuation v_i, but not that of his rival, is

$$EU(x_i, x_j; v_i) = \int_0^{x_j^{-1}(x_i)} \left[v_i - x_j(v_j)\right] \exp(-v_j)\, dv_j + \int_{x_j^{-1}(x_i)}^{\infty} x_i \exp(-v_j)\, dv_j$$

Next, apply step (2) by differentiating with respect to x_i and setting the marginal expected payoff equal to zero, and then apply step (3) by letting $x_i(v) = x_j(v) = x(v)$ to obtain $x'(v) = 2x(v) - v$. The solution to this first-order ordinary differential equation is

$$x(v) = K \exp(2v) + \frac{v}{2} + \frac{1}{4} \tag{7}$$

Turning to step (4), note that $x(v)$ is strictly increasing for all $K \geqslant 0$, so there is a continuum of such candidate equilibria with $K \geqslant 0$.

Finally, turning to step (5), note first that the support of the random variable v_i is $[0, \infty)$. Consequently for any $K \geqslant 0$, the rival's bid $x_j \in [K + \frac{1}{4}, \infty)$ in the putative equilibrium. This implies that player i cannot gain by deviating to a bid below $K + \frac{1}{4}$ since doing so yields player i a payoff of $x_i < K + \frac{1}{4}$ with probability one, which is dominated by bidding $x_i = K + \frac{1}{4}$ and earning a payoff of $K + \frac{1}{4}$ (with probability one the rival suitor's valuation is strictly positive and hence $x_j > K + \frac{1}{4}$ with probability one). If both suitors play the putative equilibrium strategies, then the expected payoff to a suitor whose valuation of the maiden is v_i is

$$EU\big(x(v_i), x(v_j); v_i\big) = K + v_i - \frac{3}{4} + \exp(-v_i)$$

[7] Results similar to those described below obtain for other distributions, including the unit Pareto distribution where $G(v) = 1 - v^{-1}$ on $[1, \infty)$ and $E[v] = \infty$.

where $K \geqslant 0$. It is routine matter to establish that there is no incentive for player i to deviate to an alternative bid $x_i \in [K + \frac{1}{4}, \infty)$; see Baye et al. (2010).

Thus, applying the usual reasoning—steps (1) through (5) in this case—one would conclude that the strategies identified in Eq. (7) comprise a symmetric Bayesian–Nash equilibrium. Since steps (1) through (5) hold for any $K \geqslant 0$, there is a continuum of symmetric pure-strategy equilibria in which a player whose valuation is $v_i \in [0, \infty)$ earns a (finite) expected payoff of $EU = K + v_i - \frac{3}{4} + \exp(-v_i) \in [v_i - \frac{3}{4} + \exp(-v_i), \infty)$. This is the Herodotus paradox with incomplete information: Even though all moments of the value distribution are bounded, and conditional on values any payment by one suitor is a pure transfer to the other, application of steps (1) through (5) leads to the conclusion that players can earn an arbitrarily high (but finite) expected payoff in a symmetric pure-strategy equilibrium.

The equilibrium strategies that obtain from steps (1) through (5), and which are summarized for the case at hand in Eq. (7), correspond to an *interim* equilibrium in that each suitor knows his own valuation of the maidens, but not the valuation of the rival suitor. Expressed differently, the paradoxical equilibrium corresponds to a situation where the two maidens are unveiled, and each suitor knows his own valuation of the most beautiful maiden but not how much the rival suitor values her. Once the maidens are unveiled (that is, conditional on each player's private information), there is a well-defined (interim) equilibrium in which each player can earn an arbitrarily high (interim) expected payoff.

Given the prospect of achieving such a blissful state once the maidens are unveiled, would the two players have an incentive to attend the auction in the first place to learn the private information required to play the interim equilibrium strategies? That is, do the putative strategies comprise an *ex ante* equilibrium? The answer is that when $K > 0$ the suitors' (*ex ante*) payoffs suffer from the same integrability problem that arose in the case of complete information: $E[X^+] \equiv E[\max(u_i, 0)] = \infty$ and $E[X^-] \equiv E[\max(-u_i, 0)] = \infty$ (see Baye et al., 2010 for details). That is, for $K > 0$, *ex ante* expected utility does not exist and hence they are incapable of determining whether or not to participate in the auction prior to the maidens being unveiled. Thus, the symmetric *ex ante* Bayesian–Nash equilibrium is unique and given by Eq. (7) with $K = 0$.

It is important to stress that the nonexistence of *ex ante* expected utility for $K > 0$ does not stem from the nonexistence of interim expected utility; indeed for any value $v_i \in [0, \infty)$ observed when the maidens are unveiled, each player's expected (interim) payoff is finite and given by $EU(v_i) = K + v_i - 3/4 + \exp(-v_i)$. Nor does nonexistence stem from pathological properties of the value distribution; all moments of the value distribution are bounded (and in fact, $E[v_i] = 1$). And, although each player's expected bid is infinite when $K > 0$, this is not sufficient for the failure of interim equilibria to be *ex ante* equilibria. Indeed, in an incomplete information variant of the Babylonian bridal auction with a matching dowry, there exist continua of both *interim* and *ex ante* symmetric equilibria even though each player's expected bid is unbounded; see Baye et al. (2010).

The analysis thus highlights that the standard methods used to derive symmetric pure-strategy equilibria in auctions with incomplete information, in fact, yield *interim* equilibria. Indeed, there are a number of important games of incomplete information in economics where steps (1) through (5) lead to unbounded bid functions, including the war of attrition (cf. Bishop et al., 1978; Riley, 1980) and the sad loser auction (Riley and Samuelson, 1981). However, it is straightforward to show that integrability conditions hold for these two games, and thus the interim equilibria are also *ex ante* equilibria.

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