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Notes

The number of active bidders in internet auctions ☆

Laurens de Haan [a,b,1], Casper G. de Vries [a,c], Chen Zhou [d,a,*,2]

[a] *Erasmus University Rotterdam, The Netherlands*
[b] *University of Lisbon, Portugal*
[c] *Chapman University, United States*
[d] *De Nederlandsche Bank, The Netherlands*

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Abstract

Internet auctions attract numerous agents, but only a few become active bidders. Under the Independent Private Values Paradigm the valuations of the active bidders form a specific record sequence. This record sequence implies that if the number n of potential bidders is large, the number of active bidders is approximately $2\log n$, potentially explaining the relative inactivity. Moreover, if the arrival of potential bidders forms a non-homogeneous Poisson process due to a time preference for auctions that are soon to close, then the arrival process of the active bidders is approximately a Poisson arrival process.

JEL classification: D44

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* Corresponding author at: Economics and Research Division, De Nederlandsche Bank, P.O. Box 98, 1000 AB, Amsterdam, The Netherlands. Fax: +31 20 5242506.

E-mail addresses: ldehaan@ese.eur.nl (L. de Haan), cdevries@ese.eur.nl (C.G. de Vries), zhou@ese.eur.nl, c.zhou@dnb.nl (C. Zhou).

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1. Introduction

Internet auctions (IA) have rapidly become a highly popular mechanism of exchange. This has created an easily accessible nation-wide or global market for rare items and standard durables, such as notebooks. It seems intuitive that a seller is better off the larger the extent of the market. Indeed, [1] showed that under the hypothesis of the Independent Private Values Paradigm (IPVP), the seller is better off by enlarging the market.[3] Similarly, from the buyer's perspective a larger choice of items and sellers is often better.

One of the perhaps surprising facts of IA markets is the low number of *active bidders*, notwithstanding the popularity of the mechanism. Consider the well known [3] study of laptop and antiques auctions on eBay and Amazon, which covers 120 eBay laptop auctions with 740 active bidders in total and 120 Amazon laptop auctions with a total of 595 active bidders. It implies an average number of 6.17 active bidders per auction on eBay and 4.96 active bidders per auction on Amazon. The [4] study of eBay coin auctions finds 3.08 active bidders on average, with a standard deviation of 2.51 and a maximum of 14 bidders. As another example, the study of the online English auction with fixed ending time in Korea by [5] reports an average number of 5.80 bidders and 8.4 bids per auction.

We provide an explanation for the low average number of active bidders relative to the potential extent of the market. Consider the number of potential bidders n as the extent of the market. Here, the *potential bidders* are those bidders who check the auction website with or without placing a bid. They are in principle interested in buying the item, but may not be willing to pay the going price. Under the IPVP we show that, given a mild identification assumption, the valuations of the active bidders form a specific record sequence. By using the probability theory of records, we then prove that if the number n of potential bidders is large, the number of active bidders is approximately equal to $2\log n$. This explains the relative inactivity, since $(2\log n)/n \to 0$ as $n \to \infty$. With this relationship in hand one can address questions such as by how much the extent of the market has increased through the creation of IA.

The $2\log n$ rule would lend itself easily to empirical scrutiny if the numbers of active and potential bidders were known. A potential proxy for the number of potential bidders is the number of page views of an IA. Unfortunately, the large IA sites such as eBay and formerly Yahoo! do not openly record the number of page views.[4]

Although no direct information on the number of potential bidders is available, information on the bid arrival time sequence is usually publicly recorded. Suppose that the potential bidders arrive according to a non-homogeneous Poisson process, where the non-homogeneity arises from the preference for auctions with short remainder time. Then we show that the active bidders arrivals approximately follow a homogeneous Poisson arrival process as a consequence of the record sequence character of their valuations.

The paper is organized as follows. In Section 2, we discuss the identification problem for IA and the maintained hypothesis. Section 3 gives our main theorem yielding the $2\log n$ rule. Section 4 investigates the arrival process of the active bidders. Section 5 concludes. Proofs are relegated to Appendix A. An example clarifying some notation is given in Appendix B.

[3] In multi-unit second price auctions entry can lead to lower revenues, see [2].

[4] Sometimes a seller can decide to include a page view counter for personal information. In a recent sample of 194 of such eBay auctions, the median number of active bidders is 10, but the median of potential bidders measured by the number of page views is 246, see [6].

2. Identification of internet auctions

An important question in the empirical analysis of auctions is whether one can retrieve the distribution of the bidders' valuation from the observed bids. The bidding system and the termination rule of an IA make the identification problem different from that for offline auctions. We first discuss these two features and then introduce our maintained hypothesis to enable identification.

IA sites usually permit a choice between alternative bid procedures. Most common is the choice between *manual* and *proxy bidding*. For a manual bid, the bidder just enters an amount higher than the currently prevailing price. The system immediately places the bid at the amount which is entered. Hence, manual bidding is like the first price open ascending bid in an English auction. A proxy bidder (secretly) communicates the maximum amount he is willing to bid to the server of the auction site, after which the machine takes over the bidding for this bidder. The proxy bidding procedure captures the second price sealed bid mechanism studied by [7]. If a newly entered manual bid is below the maximum willingness to pay of one previous proxy bidders, the system will raise the price to the minimum increment above the newly entered manual bid. Otherwise the manual bid becomes the currently prevailing price. Similarly, a new proxy bidder may find that he is immediately outbid by another proxy bidder. When two proxy bids are placed, the current price will automatically jump to the lower of the two maximum willingness-to-bid submissions plus the smallest possible increment.

There exist broadly two alternative termination rules. Either the auction ends after a pre-announced fixed lapse of time, or there is variable termination time. On eBay, the auctions have a fixed ending time, which we refer to as an "eBay-type auction". The winner is the highest bidder at the time of the close. Per contrast, when Amazon was running IAs, it provided an "Amazon-type auction" using a termination rule with an auto-extension. According to this rule, before the start of the auction an initial ending time is announced. If no bidding takes place during the last ten minutes, the auction stops at the initial pre-announced time. But if there are some bids in the last ten minutes, the ending time is automatically extended by another ten minutes. This rule also applies to the new extension period. Thus the auction only ends once no bidding activity occurs within the last ten minutes. On the former Yahoo! auction site the sellers could choose between the two termination rules.

The influence of the different termination systems on the bidding strategy is considerable. The fixed ending time on eBay-type auctions invites strategic last minute bidding, called sniping. This avoids competition, since other bidders may be unable to respond due to a lack of time, see [3]. In the Amazon-type auction sniping is not observed.

The identification question asks whether it is possible to identify the distribution of the bidder's valuation from the empirical observations of the bids; see e.g. [8]. With the specific bidding system, the number of potential bidders in an IA is unknown. Without knowing the number of bidders, [9] discusses the identification problem of IA and shows that under the IPVP it is impossible to identify the distribution of the bidder's valuations from the empirical distribution of bids, if only the second largest order statistic is observed. The parent distribution, though, is identified, when at least two order statistics are observed, i.e. the second and third largest order statistics among all valuations. Therefore, we need a maintained hypothesis in order to guarantee the identification.

To analyze the identification problem for IAs, first consider a hypothetical IA for which only the proxy bidding mechanism is available and to which the Amazon-type termination rule applies. Under the IPVP the weakly dominant strategy is to bid one's valuation. If bidders act in this way, each active bidder's valuation will be observed as his last bid, except for the winner. Next, consider the hybrid IA with the possibility of manual bidding added to the proxy bidding mechanism. To ensure that the third largest and lower valuations are observed as actual bids in the case with manual bidding, we assume that a manual bidder immediately counters with a higher bid as soon as he is overbid and as long as his valuation is above the current price. With this assumption, when the third highest valuation bidder is overbid, he immediately counters by increasing his bid. This ensures that his valuation is observed as the second highest bid. Hence, our maintained hypothesis is:

Assumption 2.1. Each active (manual) bidder immediately returns to the IA and increases his bid as soon as he is overbid and as long as his valuation is above the prevailing price.

We already noted that Assumption 2.1 is automatically satisfied when there are only proxy bidders present. For manual bidders, when being overbid, the typical IA system immediately sends a reminder enabling an immediate counter-bid. Nevertheless, with a fixed ending time, they may have strategic reasons for bidding late. Only under the Amazon-type auction termination rule, the active bidders of either type have no incentive to wait. To summarize, our assumption provides a lower bound to the number of active bidder since without an immediate response, some other manual bids might intervene.

Given Assumption 2.1, the currently prevailing price must be equal to the second highest valuation among all the potential bidders who visited the IA site. Hence, in order to motivate a new potential bidder to bid, his valuation must be higher than the prevailing second highest valuation. This conclusion is summarized in the first proposition.

Proposition 2.1. *Consider an IA with a hybrid system of manual and proxy bids. Suppose that Assumption 2.1 within the IPVP setting applies, then each active bidder's valuation is the highest or second-highest among all the valuations of the potential bidders who have visited the IA site.*

3. Main theorem

Proposition 2.1 implies that the bids can be viewed as a specific record sequence of the valuations of the potential bidders. There exists a well developed theory of records in probability theory, see [10]. This theory is used to derive the novel $2\log n$ rule. We first introduce some notation. Let $i = 1, 2, \dots, n$ denote the order in which the n potential bidders check the auction site. Suppose the valuations of all potential bidders are i.i.d. random variables $X_1, X_2, \dots, X_n$ with distribution function $F(x)$. Define the rank sequence $\{R_i\}_{i=1}^n$ as

$$R_i := \sum_{k=1}^{i} 1_{\{X_k \geqslant X_i\}},$$

where R_i is the rank of the valuation of the i-th potential bidder among the valuations of all the potential bidders who checked the auction before agent i. The valuation X_i is called a *record* if $R_i = 1$. Similarly, it is a *second record* if $R_i = 2$.

According to Proposition 2.1, a potential bidder is an active bidder if and only if $R_i \leqslant 2$. Denote the indices of active bidders as $\{J(j)\}_{j=1}^m$, where

$$J(1) = 1, \qquad J(2) = 2,$$
$$J(j+1) = \min\{i > J(j):\ R_i \leqslant 2\}, \quad j = 2, 3, \dots, m-1.$$

Here m is the number such that $R_i > 2$ for all $i > J(m)$, i.e. m is the number of active bidders. Then, the active bidders' valuations constitute a specific record sequence $\{X_{J(j)}\}_{j=1}^{m}$, dubbed the *duo-record sequence*. Note that $J(m)$ is the number of potential bidders when the m-th active bidder becomes active. An example to clarify the notation is presented in Appendix B.

Our main theorem studies the property of the index sequence $\{J(j)\}$. The proof is relegated to Appendix A.

Theorem 3.1. *As the number of potential bidders goes to infinity, the number of active bidders goes to infinity as well. Given that $k \to \infty$, the sequence*

$$\left\{\log J(k+j) - \log J(k+j-1)\right\}_{j=1}^{\infty}$$

converges in distribution to a sequence of i.i.d. exponentially distributed random variables with mean $1/2$.

Remark 3.1. In fact, based on the duo-record sequence, one can prove – in a way analogous to the proof of Corollary 4.5 in [10] for the record sequence – that the point process with points

$$\left\{\frac{1}{2}\log J(j) - \frac{1}{2}\log n\right\}_{j=1}^{\infty}$$

converges to a homogeneous Poisson point process. However, our proof of Theorem 3.1 is not based on point processes, and follows a simpler and novel approach.

Theorem 3.1 states that the differences of the sequence $\log J(j)$ are asymptotically i.i.d. and have an exponential distribution with mean $1/2$. This is related to the asymptotic behavior of $J(m)$. To obtain that, we first introduce two sequences of random variables:

$$\xi_i = 1_{\{R_i \leqslant 2\}}, \quad \text{and} \quad N(i) = \sum_{k=1}^{i} \xi_k.$$

The number $\{\xi_i\}$ indicates whether the i-th potential bidder is an active bidder or not. The sequence $\{N(i)\}$ gives the number of active bidders among the first i potential bidders. An example of these two other sequences is again given in Appendix B. The asymptotic normality of the $N(n)$ and $J(m)$ sequences is given in the following lemmas. The proofs are in Appendix A.

Lemma 3.1. *The sequence*

$$\frac{N(n) - 2\log n}{\sqrt{2\log n}}$$

is asymptotically standard normally distributed, as $n \to \infty$.

Lemma 3.2. *The sequence*

$$\frac{2\log J(m) - m}{\sqrt{m}}$$

is also asymptotically standard normally distributed, as $m \to \infty$.

Table 1
The number of potential and active bidders.

Number of potential bidders (n)	10 000	5000	1000	500	100
Estimates of active bidders ($2\log n$)	18.42	17.03	13.81	12.43	9.21
Standard deviation ($\sqrt{2\log n}$)	4.29	4.13	3.72	3.53	3.03

Note: This table gives the expected numbers of active bidders given different numbers of potential bidders. The point estimates are calculated based on the $2\log n$ rule and reported in the second row, while the standard deviations are reported in the third row.

These lemmas imply the following convergence in probability results. As $n \to \infty$ and/or $m \to \infty$, we have that

$$\frac{N(n)}{2\log n} \xrightarrow{P} 1 \quad \text{and} \quad \frac{2\log J(m)}{m} \xrightarrow{P} 1. \tag{1}$$

Note that m active bidders will be observed until there are $J(m)$ potential bidders. Conversely, there are $N(n)$ active bidders among the first n potential bidders. Therefore, (1) gives the asymptotic relationship between the numbers of potential bidders and active bidders amounting to the following $2\log n$ rule.

Rule 3.1 *($2\log n$ rule).* If the number of potential bidders n is large, the number of active bidders is approximately equal to $2\log n$.

The intuition behind the $2\log n$ rule can be grasped from the following reasoning.[5] Bidders may arrive in any order. Therefore, the probability that the n-th potential bidder has a valuation which is ranked first or second among the first n potential bidders is $2/n$. Hence, the expectation of the number of active bidders among the first n potential bidders is twice the n-th harmonic number minus one. This is approximately equal to $2\log n$ for large n.

The $2\log n$ rule relates the extent of the IA as measured by n to the market activity as measured by the number of active bidders $N(n)$. In Table 1 we relate some given numbers of potential bidders to the corresponding numbers of active bidders. The table shows that our rule implies that the number of active bidders hardly ever exceeds 20. Although the number of active bidders is relatively small, it does not mean that the extent of the IA market is small. In other words, the $2\log n$ rule illustrates that the extent of market is considerable, and has possibly greatly benefited from the fact that these items can now be sold through the internet, facilitating national and even international reach.

4. The arrivals of active bidders

The $2\log n$ rule provides a theoretical explanation for the low number of active bidders. Due to the lack of information on the number of potential bidders, however, it may be difficult to investigate the $2\log n$ rule directly. But IA sites do provide detailed records on all bids, from which one may obtain the arrival time of all active bidders. We further study the arrival process of active bidders based on the fact that their valuations form a duo-record sequence of potential bidders' valuations.

[5] We are grateful to an anonymous referee for offering this intuition.

We start with modeling the arrival process of the potential bidders. The simplest model is the homogeneous Poisson arrival process. Under Poisson arrivals, the appearance of potential bidders is random from the viewpoint of the seller and is independent from the time the auction has been running. Such an assumption appears not plausible considering the time preference of potential bidders. Although for Amazon-type auctions there is no strategic reason for late bidding, bidders may nevertheless display a preference for auctions which are close to the end of their run times in order to cut down on the time before an item is finally obtained. Most auction sites offer the possibility to easily rank the relevant auctions on the remaining time to the announced deadline.

Suppose agents actively use this feature for selecting auctions that are soon to close. Everything else equal, this preference arises from the cost of having to wait until the end of the auction before knowing whether the item is obtained. Therefore, we assume that the Poisson arrival rate λ for the arrival process of the potential bidders increases as time progresses in the following way,

$$\lambda(t) = \lambda_0 e^{\theta t}, \tag{2}$$

where θ is the time preference factor which may depend on the characteristics of the item to be auctioned. By reversing time, θ can be seen as the discount factor under continuous discounting. Given (2), the arrival process of the potential bidders is a non-homogeneous Poisson process with the instantaneous arrival rate $\lambda(t)$.

Define the time at which a record or second record occurs as the *entering time*. Given the non-homogeneous Poisson arrival process for the potential bidders, we derive the asymptotic form of the entering time process of the active bidders. The proof of the theorem is again relegated to Appendix A.

Theorem 4.1. *Suppose the potential bidders* $1, 2, \ldots, n$ *arrive at times* $T(1), T(2), \ldots, T(n)$. *Let* $\{T(i)\}_{i=1}^{\infty}$ *be the arrival times of the non-homogeneous Poisson process with a rate of occurrence function given as in* (2). *Then the arrival times of the active bidders are the sequence* $\{T(J(j))\}_{j=1}^{m}$. *For* $l \to \infty$, *the sequence* $\{T(J(l+j)) - T(J(l+j-1))\}_{j=1}^{\infty}$ *coincides asymptotically with an i.i.d. sequence with exponentially distributed innovations that have mean* $1/(2\theta)$.

Notice that when the arrival time intervals follow an i.i.d. exponential distribution, the arrival process is a homogeneous Poisson process. The result of Theorem 4.1 indicates that for (late) active bidders, their arrival process can be approximated by a homogeneous Poisson process with an arrival rate 2θ.

An intuitive explanation of Theorem 4.1 stems from the $2\log n$ rule. Following the arrival rate function $\lambda(t)$, the expected number of potential bidders that arrive before time t is $\int_0^t \lambda(t)\,dt = \lambda_0(e^{\theta t} - 1)/\theta$. This is of the order $e^{\theta t}$ for large t. Following the $2\log n$ rule, the expected number of active bidders is approximately $2\theta t$. Since this holds for any large t, we get that the late arrivals of the active bidders follow approximately a homogeneous Poisson arrival process with arrival rate 2θ. To summarize, although the arrival process of the potential bidders is a non-homogeneous process due to the time preference, the arrival process of the active bidders is a homogeneous Poisson process due to the duo-record feature.

The result of this theorem is indirectly associated with the $2\log n$ rule in Rule 3.1. Both of the results are a consequence of the duo-record theory, while the result on the arrival of active bidders requires an extra maintained assumption that the arrival of potential bidders follow a specific non-homogeneous Poisson process with the arrival rate function in (2).

5. Conclusion

Internet auctions attract numerous agents, but only a few become active bidders. The number of potential bidders is the extent of the market. This number, however, is unknown to the internet auctioneer. In this paper, we study the connection between the number of potential bidders and the number of active bidders, in order to explain the low number of active bidders.

Our study started from the bidding process of the Amazon-type IA, which is a hybrid of the English auction and the second price auction. Under the assumption that active bidders immediately respond to overbidding, the active bidders' valuations are a duo-record sequence of the potential bidders' valuation sequence.

We proved that the logarithmic difference between the indices of active bidders are asymptotically i.i.d. exponentially distributed with mean $1/2$. The number of potential bidders is thus connected with the number of active bidders through the $2 \log n$ rule.

On large IA websites, the number of potential bidders is not openly reported, which hampers empirical investigation into the $2 \log n$ rule. We therefore also model the arrival process of the active bidders, which is recorded by the IA sites. If the potential bidders come to the auction according to a non-homogeneous Poisson process due to time preference for auctions that are soon to close, then the publicly available entering time sequence of the active bidders is approximately a homogeneous Poisson process.

The $2 \log n$ rule explains why there are usually few active bidders. Since the number of active bidders is logarithmically related to the number of bidders potentially showing interest for a particular IA, the extent of the market can be much larger than is revealed by direct observation of the active bidders. With 10 000 potential bidders, 10 active bidders are within the 95% confidence band, but 25 active bidders are also compatible. Thus the number of active bidders may at times be quite different given the number of potential bidders. Conversely, given the number of active bidders, the extent of the market may vary a lot. Therefore, although the $2 \log n$ rule explains the low observed bidding activity, due to the uncertainty given in Lemmas 3.1 and 3.2, a high number of active bidders does not necessarily imply a very large number of potential bidders for a particular auction.

To see the wider economic relevance of the $2 \log n$ rule, consider a case of an IA where there are 500 potential bidders. Suppose the distribution of valuations is uniform on $[0, 1000]$. By Table 1, the expected number of active bidders is 12. If the seller were to calculate the potential gain from the auction by only considering the number of active bidders, he would arrive at $(11/13) \cdot 1000\$ = 846.15\$$. This would be a considerable underestimate of the true gains. Since there are 500 potential bidders, the expected revenue is in fact $499/501 \cdot 1000\$ = 996.01\$$. This is about 150$ higher than the back of an envelope calculated guesstimate on basis of the directly observed active number of bidders.

Appendix A

Proof of Lemma 3.1. From the independence of the $\{X_i\}_{i=1}^n$, we get that $\{R_i\}_{i=1}^n$ is a sequence of independent random variables, see [10, Proposition 4.3(i)]. Thus, the $\{\xi_i\}_{i=1}^n$ are also independent. Since $N(n)$ is the partial sum sequence of $\{\xi_i\}_{i=1}^n$, by using the central limit theorem for independent bounded random variables, we get the asymptotic normality immediately.

The asymptotic mean and variance are calculated from the distribution of the $\{\xi_i\}_{i=1}^n$. From Proposition 4.3(i) in [10], we have

$$P(R_i = s) = 1/i, \quad s = 1, 2, \ldots, i,$$

so that

$$P(\xi_i = 1) = \frac{2}{i} \quad \text{and} \quad P(\xi_i = 0) = 1 - \frac{2}{i}. \tag{3}$$

This implies

$$EN(n) = 1 + \sum_{i=2}^{n} \frac{2}{i} = 2\log n + o(\sqrt{\log n}),$$

and

$$Var\big(N(n)\big) = \sum_{i=2}^{n} \frac{2}{i} - \sum_{i=2}^{n} \frac{4}{i^2} \sim 2\log n.$$

This proves the lemma. □

Proof of Lemma 3.2. For fixed positive number x, denote

$$n_0(m,x) = \left[\exp\left(\frac{xm^{1/2} + m}{2}\right)\right].$$

Then,

$$\frac{m - 2\log n_0}{\sqrt{2\log n_0}} \to -x \quad \text{as } m \to \infty.$$

Since $J(m)$ is an integer, we have

$$P\left(\frac{2\log J(m) - m}{\sqrt{m}} \leqslant x\right) = P\big(J(m) \leqslant n_0(m,x)\big).$$

Notice that the two events $\{J(m) \leqslant n_0\}$ and $\{N(n_0) \geqslant m\}$ are actually the same. Therefore,

$$\begin{aligned} P\left(\frac{2\log J(m) - m}{\sqrt{m}} \leqslant x\right) &= P\big(J(m) \leqslant n_0\big) \\ &= P\big(N(n_0) \geqslant m\big) \\ &= P\left(\frac{N(n_0) - 2\log n_0}{\sqrt{2\log n_0}} \geqslant \frac{m - 2\log n_0}{\sqrt{2\log n_0}}\right) \\ &\to 1 - \Phi(-x) = \Phi(x) \quad \text{as } m \to \infty. \end{aligned}$$

Here we use Lemma 3.1 in the last step. This proves Lemma 3.2. □

Proof of Theorem 3.1. As a consequence of Lemma 3.1, when $n \to \infty$, $m = N(n) \to \infty$. Similarly, from Lemma 3.2, $J(m) \xrightarrow{P} \infty$ when $m \to \infty$.

As we discussed in the proof of Lemma 3.1, the random variables $\{\xi_i\}_{i=1}^{n}$ are independent, and the distribution is given in (3). So we have

$$\begin{aligned} & P\big(J(j+1) > s \mid J(j) = s_n, J(j-1) = s_{n-1}, \ldots, J(1) = s_1\big) \\ &= P(\xi_{s_n+1} = 0, \ldots, \xi_{s-1} = 0, \xi_s = 0) \\ &= \left(1 - \frac{2}{s_n+1}\right)\left(1 - \frac{2}{s_n+2}\right)\cdots\left(1 - \frac{2}{s-1}\right)\left(1 - \frac{2}{s}\right) \end{aligned}$$

$$= \frac{s_n - 1}{s_n + 1} \cdot \frac{s_n}{s_n + 2} \cdot \frac{s_n + 1}{s_n + 3} \cdots \frac{s - 3}{s - 1} \cdot \frac{s - 2}{s}$$
$$= \frac{s_n(s_n - 1)}{s(s - 1)} \tag{4}$$

which implies that $\{J(j)\}_{j=1}^n$ is a Markov process. From (4), subsequently,

$$P\big(\log J(j+1) - \log J(j) > x \,\big|\, J(j) = s_n, \dots, J(1) = s_1\big)$$
$$= P\big(J(j+1) > e^x s_n \,\big|\, J(j) = s_n, J(j-1) = s_{n-1}, \dots, J(1) = s_1\big)$$
$$= \frac{s_n(s_n - 1)}{[e^x s_n]([e^x s_n] - 1)}$$
$$\to e^{-2x}, \quad \text{as } s_n \to \infty.$$

Combining with the fact that $J(m) \xrightarrow{P} \infty$ as $m \to +\infty$, the theorem is proved. □

Proof of Theorem 4.1. Let $M(t)$ be the number of potential bidders arriving at the auction site before time t. According to the property of a non-homogeneous Poisson process, $M(t)$ follows a Poisson distribution with mean

$$\mu(t) = \int_0^t \lambda(s)\,ds = \int_0^t \lambda_0 e^{\theta s}\,ds = \lambda_0 \frac{e^{\theta t} - 1}{\theta}.$$

Hence, as $t \to \infty$, $\mu(t) \to \infty$ and

$$\frac{\mu(t)}{e^{\theta t}} \to \frac{\lambda_0}{\theta} =: c.$$

Consider the family of random variables $\{M(t)/\mu(t)\}$. Notice that as $t \to \infty$,

$$Var\big(M(t)/\mu(t)\big) = Var\big(M(t)\big)/\big(\mu(t)\big)^2 = 1/\mu(t) \to 0.$$

So we get that $M(t)/\mu(t) \to 1$ in probability. Therefore, as $t \to \infty$

$$\frac{M(t)}{e^{\theta t}} \xrightarrow{P} c,$$

which implies that $\log M(t) - \theta t \to \log c$ in probability.

Replace t with $T(J(l+j))$ and let $l \to \infty$. Considering the fact that $M(T(J(l+j))) = J(l+j)$, we get that

$$\log J(l+j) - \theta T\big(J(l+j)\big) \xrightarrow{P} \log c.$$

Replacing j with $j-1$ in this relation, we also have that

$$\log J(l+j-1) - \theta T\big(J(l+j-1)\big) \xrightarrow{P} \log c.$$

Combining these limit relations, we get that

$$\big(\log J(l+j) - \log J(l+j-1)\big) - \theta\big(T\big(J(l+j)\big) - T\big(J(l+j-1)\big)\big) \xrightarrow{P} 0.$$

By the conclusion of Theorem 3.1, Theorem 4.1 follows. □

Table 2
Example for notations.

i	1	2	3	4	5	6
X_i	30	10	60	20	50	40
R_i	1	2	1	3	2	3
ξ_i	1	1	1	0	1	0
$N(i)$	1	2	3	3	4	4
j	1	2	3	–	4	–
$J(j)$	1	2	3	–	5	–
Price	1	11	31	–	51	–

Note: This table gives an illustration of the notations we used in the paper. The index i indicates the arrival sequence of the potential bidders. The variable X_i gives the private value of each potential bidder. The rank sequence R_i is the rank of the valuation of the i-th potential bidder among the valuations of the first i potential bidders. The indicator ξ_i indicates whether potential bidder i is an active bidder or not. The sequence $N(i)$ gives the number of active bidders among the first i potential bidders. The index j indicates the j-th active bidder, while $J(j)$ indicates its corresponding index among the potential bidders. The last row reports the prevailing price up to the arrival of the i-th potential bidder.

Appendix B

Here we present an example to clarify the notation in the paper. Suppose we have the index sequence $i = 1, 2, \ldots, 6$ with the valuation sequence $X_1 = 30$, $X_2 = 10$, $X_3 = 60$, $X_4 = 20$, $X_5 = 50$, $X_6 = 40$. Moreover, suppose all bids are proxy bids and that 1 is the minimum bid increment. In Table 2, we present the rank sequence R_i, the indicator sequence ξ_i, and the number of active bidder sequence $N(i)$. In this example, the corresponding index sequence $J(j)$ is given as $J(1) = 1$, $J(2) = 2$, $J(3) = 3$, $J(4) = 5$. In this case, $m = 4$.

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